

SEMIAMPLE PERTURBATIONS FOR LOG CANONICAL VARIETIES OVER AN F -FINITE FIELD CONTAINING AN INFINITE PERFECT FIELD

HIROMU TANAKA

ABSTRACT. Let k be an F -finite field containing an infinite perfect field of positive characteristic. Let (X, Δ) be a projective log canonical pair over k . In this note we show that, for a semi-ample divisor D on X , there exists an effective \mathbb{Q} -divisor $\Delta' \sim_{\mathbb{Q}} \Delta + D$ such that (X, Δ') is log canonical if there exists a log resolution of (X, Δ) .

1. MAIN THEOREM

In this note, we prove the following theorem that is non-trivial even for the klt case.

Theorem 1. *Fix $\mathbb{K} \in \{\mathbb{Q}, \mathbb{R}\}$. Let k be an F -finite field containing an infinite perfect field k_0 of characteristic $p > 0$. Let (X, Δ) be a projective log canonical (resp. klt) pair over k , where Δ is an effective \mathbb{K} -divisor. Let D be a semi-ample \mathbb{K} -Cartier \mathbb{K} -divisor on X . If there is a log resolution of (X, Δ) , then there exists an effective \mathbb{K} -Cartier \mathbb{K} -divisor $D' \sim_{\mathbb{K}} D$ such that $(X, \Delta + D')$ is log canonical (resp. klt).*

In characteristic zero, Theorem 1 holds by Bertini's theorem for free divisors, which fails in positive characteristic. Our proof depends on the theory of F -singularities. More precisely, Theorem 1 follows from Proposition 2 and Proposition 3.

Proposition 2. *Fix $\mathbb{K} \in \{\mathbb{Q}, \mathbb{R}\}$. Let k be an F -finite field containing an infinite perfect field k_0 of characteristic $p > 0$. Let (X, Δ) be a projective strongly F -regular variety over k , where Δ is an effective \mathbb{K} -divisor. Let D be a semi-ample \mathbb{K} -divisor on X . If X is regular, then there exists an effective \mathbb{K} -divisor $D' \sim_{\mathbb{K}} D$ such that $(X, \Delta + D')$ is strongly F -regular.*

Proposition 3. *Let k be an F -finite field containing an infinite perfect field k_0 of characteristic $p > 0$. Let X be a projective regular variety*

2010 Mathematics Subject Classification. 14E30, 13A35.

Key words and phrases. log canonical, semi-ample, positive characteristic.

over k and let Δ be an effective simple normal crossing \mathbb{Q} -divisor on X whose coefficients are contained in $[0, 1]$. Let D be a semi-ample \mathbb{Q} -divisor on X . Then there exists an effective \mathbb{Q} -divisor $D' \sim_{\mathbb{Q}} D$ such that $(X, \Delta + D')$ is sharply F -pure.

Idea of Theorem 1: We overview the proof of Theorem 1. We only treat the case $\mathbb{K} = \mathbb{Q}$, (X, Δ) is klt, and k is an infinite perfect field.

Since there is a log resolution of (X, Δ) , we may assume that X is smooth and Δ is simple normal crossing. In this case, the notions of klt and strongly F -regular singularities coincide, hence it suffices to show Proposition 2 because strongly F -regular singularities are klt.

We may assume that D is base point free. By [PSZ, Theorem B], we can find a large integer $m \in \mathbb{Z}_{>0}$ such that

$$(X, \Delta + \frac{1}{m}(D_1 + \cdots + D_{\dim X}))$$

is strongly F -regular for every $D_i \in |D|$. Since D is base point free, we can find members

$$D_1, \dots, D_m \in |D|$$

which satisfy the following property:

(*) the intersection $\bigcap_{j \in J} D_j$ is empty for every $|J| = \dim X + 1$.

Set

$$D' := \frac{1}{m}(D_1 + \cdots + D_m) \sim_{\mathbb{Q}} D.$$

By (*), for every point $x \in X$, there is an open subset $U \subset X$ such that

$$(U, (\Delta + D')|_U) = (U, (\Delta + \frac{1}{m} \sum_{i \in I} D_i)|_U)$$

for some $I \subset \{1, 2, \dots, m\}$ with $0 \leq |I| \leq \dim X$. Therefore, $(X, \Delta + D')$ is strongly F -regular.

If k is not a perfect field, then we need to replace [PSZ, Theorem B] in the above argument with Proposition 8. Although the proofs of Proposition 2 and Proposition 3 are very similar, the proof of Proposition 3 needs the inversion of adjunction for F -singularities established by [Schwede].

Assumption on the base field: Let us consider about the assumption that k contains an infinite perfect field k_0 . First, we impose a restrictive condition: k is perfect. Then, our assumption $k_0 \subset k$ is equivalent to the condition that k is an infinite perfect field. In this case, a key result (Proposition 8) almost follows from [PSZ, Theorem B], however we need the arguments in the proofs of Proposition 2

and Proposition 3. For the case when k is a finite field, our proof encounters a similar technical difficulty to the Bertini theorem for very ample divisors. Although the author does not know whether Theorem 1 holds for finite fields, the method of [Poonen] may help us.

Second, let us go back to the general situation. In the proof of a key result (Proposition 8), we will make use of rational points or its p^e -powers of \mathbb{A}_k^r . To assure the existence of infinitely many p^∞ -torsion points, we assume that k contains an infinite perfect field.

Motivation: Originally the motivation for Theorem 1 is to show the relative klt/lc abundance theorem assuming the absolute abundance theorem. For example, given a morphism $\pi : (X, \Delta) \rightarrow S$ from a projective klt variety (X, Δ) to a normal variety S , if $K_X + \Delta$ is f -nef, then we see that $K_X + \Delta + \pi^* A_S$ is nef for some ample divisor A_S up to the cone theorem. Since $\pi^* A_S$ is semi-ample, Theorem 1 implies that there is an effective \mathbb{Q} -divisor $\Delta' \sim_{\mathbb{Q}} \Delta + \pi^* A_S$ such that (X, Δ') is klt and $K_X + \Delta$ is nef. Thus, we could reduce the problem to the absolute case. In this situation, our assumption on the base field k is harmless because we can assume this as follows: first take the base change to the composite field of k and $\overline{\mathbb{F}}_p$, and second we can find a subfield of k which is finitely generated over $\overline{\mathbb{F}}_p$ and defines all the given varieties and morphisms.

Related results: Recently, some problems on birational geometry are solved by the theory of F -singularities ([CHMS], [CTX], [HX], [Mustață]). By using it, also this paper shows a result on birational geometry (Theorem 1). For some related topics of F -singularities, see [BSTZ], [BST]. In particular, Bertini's theorem of very ample divisors holds for F -singularities ([SZ]).

Acknowledgments. The author would like to thank Professors Paolo Cascini, Mircea Mustață, Zsolt Patakfalvi, Karl Schwede and Shunsuke Takagi for very useful comments and discussions.

2. NOTATION

We say X is a *variety* over a field k (or k -variety) if X is an integral scheme which is separated and of finite type over k .

Let X be a normal variety over a field k . For a closed subset Z of X , we say Z is a *simple normal crossing divisor* if, for the irreducible decomposition $Z = \bigcup_{i \in I} D_i$, we obtain $\dim D_i = \dim X - 1$ for every $i \in I$ and $\bigcap_{j \in J} D_j$ is regular for every subset $\emptyset \neq J \subset I$, where we consider D_i as the reduced scheme and the intersection $\bigcap_{j \in J} D_j$ means a scheme-theoretic intersection.

Let X be a normal k -variety and let D be an effective \mathbb{R} -divisor. We say $\mu : W \rightarrow X$ is a *log resolution* of (X, D) if μ is a projective birational morphism, W is regular and $\mu^{-1}(D) \cup \text{Ex}(\mu)$ is a simple normal crossing divisor.

We will freely use the notation and terminology in [Kollár]. In particular, for the definitions of klt and log canonical singularities, see [Kollár, Definition 2.8].

For the definitions of strongly F -regular and sharply F -pure pairs, see [Schwede, Definition 2.7] and [CTX, Definition 2.7]. For $e \in \mathbb{Z}_{>0}$, a normal variety X and an effective \mathbb{Z} -divisor D on X , we have a *trace map*

$$\text{Tr}_X^e(D) : F_*^e(\mathcal{O}_X(-(p^e - 1)K_X - D)) \rightarrow \mathcal{O}_X.$$

For the definition and the basic properties of trace map, see [CTX, Section 2.3].

We say an \mathbb{R} -Cartier \mathbb{R} -divisor D is *semi-ample* if we can write $D = \sum_{1 \leq i \leq r} a_i D_i$, where $a_i \in \mathbb{R}_{\geq 0}$ and D_i is a semi-ample Cartier divisor.

3. PROOFS

We recall a basic lemma.

Lemma 4. *Let A be an integral domain of characteristic $p > 0$ and assume that A has a p -basis $\{x_1, \dots, x_n\}$. Then, $\{x_1, \dots, x_n, t_1, \dots, t_r\}$ is a p -basis of $A[t_1, \dots, t_r]$.*

Proof. We may assume that $r = 1$ and set $t_1 =: t$. Consider the following A -module homomorphism

$$\varphi : \bigoplus_{0 \leq i_\ell, j < p} A \cdot x_1^{i_1} \cdots x_n^{i_n} t^j \rightarrow F_*(A[t]), \quad x_1^{i_1} \cdots x_n^{i_n} t^j \mapsto F_*(x_1^{i_1} \cdots x_n^{i_n} t^j),$$

where the left hand side is the free A -module generated by a basis $\{x_1^{i_1} \cdots x_n^{i_n} t^j\}_{0 \leq i_\ell, j < p}$. The surjectivity is clear. We show the injectivity. Assume that

$$\sum_{0 \leq i_\ell, j < p} f_{I,j}(t)^p x_1^{i_1} \cdots x_n^{i_n} t^j = 0 \quad \text{in } A[t],$$

for some $f_{I,j}(t) \in A[t]$ where $I := (i_1, \dots, i_n)$ is the multi-index. We show $f_{I,j}(t) = 0$ for every I and j . We can write $f_{I,j}(t) = \sum_k a_k^{I,j} t^k$ for some $a_k^{I,j} \in A$. We obtain

$$\sum_{I,j} f_{I,j}(t)^p x_1^{i_1} \cdots x_n^{i_n} t^j = \sum_{I,j} \left(\sum_k a_k^{I,j} t^k \right)^p x_1^{i_1} \cdots x_n^{i_n} t^j = \sum_{I,j,k} (a_k^{I,j})^p x_1^{i_1} \cdots x_n^{i_n} t^{pk+j} = 0.$$

Since every coefficient of t^{pk+j} vanishes, we obtain

$$\sum_I (a_k^{I,j})^p x_1^{i_1} \cdots x_n^{i_n} = 0$$

for every j and k . By the definition of p -basis, we obtain $a_k^{I,j} = 0$ for every I, j and k . Thus $f_{I,j}(t) = \sum_k a_k^{I,j} t^k = 0$ for every I and j . We are done. \square

The following lemma essentially follows from [Schwede].

Lemma 5. *Let Y be a regular variety over an F -finite field of characteristic $p > 0$. Let S be a reduced simple normal crossing divisor and let $S = \sum_{i \in I} S_i$ be the irreducible decomposition. Set $T := \bigcap_{i \in I} S_i$. Let E be an effective Cartier divisor on X such that $T \not\subset \text{Supp} E$. Fix a (possibly non-closed) point $y \in T$ and $e \in \mathbb{Z}_{>0}$. Then, the following assertions are equivalent.*

- (1) $\text{Tr}_Y^e((p^e - 1)S + E)$ is surjective at y .
- (2) $\text{Tr}_T^e(E|_T)$ is surjective at y .

Proof. By the construction of the trace map, we obtain the following commutative diagram (cf. [T, Lemma 2.6(1)]):

$$\begin{array}{ccc} F_*^e(\mathcal{O}_Y(-(p^e - 1)(K_X + S) - E)) & \xrightarrow{\text{surjection}} & F_*^e(\mathcal{O}_T(-(p^e - 1)K_T - E|_T)) \\ \downarrow \text{Tr}_Y^e((p^e - 1)S + E) & & \downarrow \text{Tr}_T^e(E|_T) \\ \mathcal{O}_Y & \xrightarrow[\text{surjection}]{\rho} & \mathcal{O}_T. \end{array}$$

If $\text{Tr}_Y^e((p^e - 1)S + E)$ is surjective at y , then so is $\text{Tr}_T^e(E|_T)$ by a diagram chase. Thus, assume that $\text{Tr}_T^e(E|_T)$ is surjective at y . By a diagram chase, there exists $a \in \text{Im}(\text{Tr}_Y^e((p^e - 1)S + E))$ such that $\rho(a) = 1 \in \mathcal{O}_{T,y}$. Then, we see $a - 1 \in I_T \mathcal{O}_{Y,y} \subset \mathfrak{m}_y \mathcal{O}_{Y,y}$. This implies that a is a unit of $\mathcal{O}_{Y,y}$. We are done. \square

Since we would like to treat rational points of \mathbb{A}_k^r and their p^e -powers, we introduce the following terminologies.

Definition 6. Let k be a field. A *coordinate* $t := [t_1, \dots, t_r]$ of \mathbb{A}_k^r is a set of elements of $\mathcal{O}_{\mathbb{A}_k^r}(\mathbb{A}_k^r)$ which satisfies $k[t_1, \dots, t_r] = \mathcal{O}_{\mathbb{A}_k^r}(\mathbb{A}_k^r)$. This induces a bijection:

$$\theta : k^r \rightarrow \mathbb{A}_k^r(k), \quad (c_1, \dots, c_r) \mapsto (t_1 - c_1, \dots, t_r - c_r).$$

For a subfield $k_0 \subset k$, we define $\mathbb{A}_k^r(t, k_0) := \theta(k_0^r)$.

Remark 7. The set $\mathbb{A}_k^r(t, k_0)$ is a subset of $\mathbb{A}_k^r(k)$, which depends on the choice of a coordinate $t = [t_1, \dots, t_r]$. Actually, for every $a \in \mathbb{A}_k^r(k)$ and every subfield $k_0 \subset k$, we can find a coordinate $t = [t_1, \dots, t_r]$ such that $a \in \mathbb{A}_k^r(t, k_0)$. On the other hand, we obtain $\mathbb{A}_k^r(t, k_0) = \mathbb{A}_k^r(t', k_0)$ for another coordinate $t' = [t'_1, \dots, t'_r]$ such that $t'_i = t_i - c_i$ with $c_i \in k_0$.

Proposition 8 is a key result in this paper, which compares a special fiber and general fibers. For the case where k is perfect, Proposition 8 is very similar to [PSZ, Theorem B].

Proposition 8. *Let k be an F -finite field of characteristic $p > 0$. Let X be a regular variety over k . We fix the followings.*

- $e \in \mathbb{Z}_{>0}$.
- $T := \mathbb{A}_k^r$.
- An effective Cartier divisor E on $X \times_k T$ such that $\text{Supp } E$ contains no fibers of the second projection $X \times_k T \rightarrow T$.
- A closed point $x \in X$ and a rational point $a \in T(k)$.
- A coordinate $t = [t_1, \dots, t_r]$ of T such that $a \in T(t, k^{p^e})$ (cf. Definition 6).

If $\text{Tr}_{X \times_k \{a\}}^e(E|_{X \times_k \{a\}})$ is surjective at (x, a) , then there exists an open subset $(x, a) \in U$ of $X \times_k T$ such that $\text{Tr}_{X \times_k \{b\}}^e(E|_{X \times_k \{b\}})$ is surjective at (y, b) for every closed point $(y, b) \in U$ with a closed point $y \in X$ and $b \in T(t, k^{p^e})$.

Proof. We may assume that

$$X = \text{Spec } R_X, \quad T = \text{Spec } k[t_1, \dots, t_r] = \text{Spec } R_T.$$

Moreover, we can assume that $a = (t_1, \dots, t_r) \in R_T$, that is, a is the origin (Remark 7). By shrinking $\text{Spec } R_X$ around x , we may assume that R_X has a p -basis $\{x_1, \dots, x_n\}$:

$$R_X = \bigoplus_{0 \leq i_\alpha < p^e} R_X^{p^e} x_1^{i_1} \cdots x_n^{i_n}.$$

Take an element $0 \neq \rho \in R_X \otimes_k R_T$ which satisfies the following properties.

- (1) $(x, a) \in \text{Spec}((R_X \otimes_k R_T)[\rho^{-1}])$.
- (2) $E|_{\text{Spec}((R_X \otimes_k R_T)[\rho^{-1}])} = \text{div}(f)$ where $f \in (R_X \otimes_k R_T)[\rho^{-1}]$.

Set

$$S := (R_X \otimes_k R_T)[\rho^{-1}].$$

Let S_i on $X \times_k T$ be the pull-back of the regular effective Cartier divisor $\text{div}(t_i)$ on T . By Proposition 5, we see that the trace map

$$\text{Tr}_{X \times T}^e((p^e - 1) \sum_{i=1}^r S_i + E)$$

is surjective at (x, a) . Shrinking $\text{Spec } S$ around (x, a) if necessary, we obtain the following splitting:

$$\text{id}_S : S \rightarrow F_*^e S \xrightarrow{\times f(t_1 \cdots t_r)^{p^e - 1}} F_*^e S \xrightarrow{\varphi} S.$$

It suffices to show the splitting of

$$S \rightarrow F_*^e S \xrightarrow{\times f((t_1 - b_1) \cdots (t_r - b_r))^{p^e - 1}} F_*^e S$$

for general $(b_1, \dots, b_r) \in (k^{p^e})^r$.

Since

$$\{x_1, \dots, x_n, t_1 - b_1, \dots, t_r - b_r\}$$

is a p -basis of $R_X \otimes_k k[t_1, \dots, t_r]$ for every $b_i \in k$ (Lemma 4), it is also a p -basis of

$$(R_X \otimes_k k[t_1, \dots, t_r])[\rho^{-1}] = (R_X \otimes_k R_T)[\rho^{-1}] = S.$$

For every $b := (b_1, \dots, b_r) \in k^r$, we obtain the following expressions

$$\begin{aligned} g &= \sum_{0 \leq i_\alpha < p^e, 0 \leq j_\beta < p^e} \xi_{I,J}^{p^e}(x_1^{i_1} \cdots x_n^{i_n} t_1^{j_1} \cdots t_r^{j_r}) \\ &= \sum_{0 \leq i_\alpha < p^e, 0 \leq j_\beta < p^e} \xi_{I,J}(b)^{p^e} (x_1^{i_1} \cdots x_n^{i_n} (t_1 - b_1)^{j_1} \cdots (t_r - b_r)^{j_r}) \end{aligned}$$

where $\xi_{I,J}, \xi_{I,J}(b) \in S$, $I := (i_1, \dots, i_n)$, and $J := (j_1, \dots, j_r)$. Clearly, $\xi_{I,J}(0) = \xi_{I,J}$. By applying the binomial expansion to $t_\ell^{j_\ell} = ((t_\ell - b_\ell) + b_\ell)^{j_\ell}$, we obtain

$$\xi_{I,0}(b)^{p^e} = \sum_{0 \leq j_\beta < p^e} \xi_{I,J}^{p^e} b_1^{j_1} \cdots b_r^{j_r}$$

for every I and $b = (b_1, \dots, b_r) \in (k^{p^e})^r$.

Consider the following polynomial

$$\eta_{I,0}(z_1, \dots, z_r) := \sum_{0 \leq j_\beta < p^e} \xi_{I,J}^{p^e} z_1^{j_1} \cdots z_r^{j_r} \in S[z_1, \dots, z_r].$$

For $b = (b_1, \dots, b_r) \in (k^{p^e})^r$, we obtain

$$\eta_{I,0}(b_1, \dots, b_r) = \sum_{0 \leq j_\beta < p^e} \xi_{I,J}^{p^e} b_1^{j_1} \cdots b_r^{j_r} = \xi_{I,0}(b)^{p^e}.$$

If $\xi_{I,0}(b) \notin \mathfrak{m}_{(y,b)}$, then $\mathrm{Tr}_{X \times \{b\}}^e(E|_{X \times_k \{b\}})$ is surjective at (y, b) . Thus it suffices to find an index I' such that $\xi_{I',0}(b) \notin \mathfrak{m}_{(y,b)}$ for every general (y, b) .

Recall

$$t_i \in k[t_1, \dots, t_r] = R_T \subset R_X \otimes_k R_T \subset (R_X \otimes_k R_T)[\rho^{-1}] = S,$$

where we write $t_i = 1 \otimes_k t_i \in S$ by abuse of notation. Thus, we obtain

$$\eta_{I,0}(t_1, \dots, t_r) \in S.$$

Then, for a closed point $x \in X$ and $b = (b_1, \dots, b_r) \in (k^{p^e})^r$ (i.e. b corresponds to the maximal ideal $(t_1 - b_1, \dots, t_r - b_r)$), the following three assertions are equivalent.

- $\eta_{I,0}(t_1, \dots, t_r) \notin \mathfrak{m}_{(y,b)}$.
- $\eta_{I,0}(b_1, \dots, b_r) \notin \mathfrak{m}_{(y,b)}$.
- $\xi_{I,0}(b) \notin \mathfrak{m}_{(y,b)}$.

Since φ gives the above splitting, we can find a multi-index $I' = (i'_1, \dots, i'_n)$ such that $\xi_{I',0} = \xi_{I',0}(0) \notin \mathfrak{m}_{(x,a)}$. This implies

$$\eta_{I',0}(t_1, \dots, t_r) \notin \mathfrak{m}_{(x,a)}.$$

Thus, we can find an open set $(x, a) \in U \subset X \times_k T$, such that

$$\eta_{I',0}(t_1, \dots, t_r) \notin \mathfrak{p}$$

for every prime ideal $\mathfrak{p} \in U$. Therefore, for every $(x, b) \in U$ where $x \in X$ is a closed point and $b \in T(k^{p^e})$, we obtain $\xi_{I',0}(b) \notin \mathfrak{m}_{(y,b)}$. \square

Proof of Proposition 2. First, we reduce the proof to the case $\mathbb{K} = \mathbb{Q}$. By enlarging coefficients of Δ , we may assume that Δ is a \mathbb{Q} -divisor (cf. [CTX, Remark 2.8(2)]). We can write $D = \sum_{1 \leq i \leq s} a_i D_i$, where $a_i \in \mathbb{R}_{\geq 0}$ and each D_i is a semi-ample Cartier divisor. By the induction on s , we may assume that $s = 1$. Thus we obtain $D = a_1 D_1$. By replacing a_1 with $\lceil a_1 \rceil$, we may assume that D is a semi-ample Cartier divisor. Thus, we could reduce the proof to the case $\mathbb{K} = \mathbb{Q}$.

From now on, we assume that $\mathbb{K} = \mathbb{Q}$ and we show the assertion in the proposition. Since X is projective, we can find an effective \mathbb{Q} -divisor H on X with the following properties.

- $(X, \Delta + H)$ is strongly F -regular.
- $\mathrm{Supp} \Delta \subset \mathrm{Supp} H$.
- $(X \setminus \mathrm{Supp} H, \Delta|_{X \setminus \mathrm{Supp} H} = 0)$ is globally F -regular.

By enlarging coefficients of Δ and H a little, we may assume that $(p^{d_0} - 1)\Delta$ and $(p^{d_0} - 1)H$ are \mathbb{Z} -divisors for some $d_0 \in \mathbb{Z}_{>0}$. By replacing D with its some multiple, we may assume that D is a Cartier divisor

such that $|D|$ is base point free. In particular, $\dim_k H^0(X, D) \geq 2$, otherwise $D \sim 0$ and the assertion is trivial.

Set $T_1 := \mathbb{A}_k^q = \mathbb{A}(H^0(X, D))$ (i.e. $q := \dim_k H^0(X, D)$) and

$$T := (T_1)^{\dim X} = \mathbb{A}_k^q \times_k \cdots \times_k \mathbb{A}_k^q \simeq \mathbb{A}_k^{q \dim X}.$$

We can find an effective Cartier divisor \mathcal{D} on $X \times_k T$ which satisfies the following properties:

- Each k -rational point $a \in (\mathbb{A}_k^q(k))^{\dim X} = T(k)$ corresponds to a pair $(D_1, \dots, D_{\dim X})$ where, for every i , we obtain $D_i \sim D$ is an effective Cartier divisor. In this case, we write $a = [D_1, \dots, D_{\dim X}]$.
- $\mathcal{D}|_{X \times_k \{[D_1, \dots, D_{\dim X}]\}} = D_1 + \cdots + D_{\dim X}$.

Fix a coordinate $t = (t_1, \dots, t_{q \dim X})$ of T . Note that we have a subset $T(t, k_0) \subset T(k)$, which is dense in T (cf. Definition 6).

We show that there exists $e_0 \in d_0 \mathbb{Z}_{>0}$ such that the trace map

$$\mathrm{Tr}_X^{e_0}((p^{e_0} - 1)(\Delta + H) + 2(D_1 + \cdots + D_{\dim X}))$$

is surjective for every $[D_1, \dots, D_{\dim X}] \in T(t, k_0)$. Fix $a = [D_1, \dots, D_{\dim X}] \in T(t, k_0)$. Then, we can find $d_a \in \mathbb{Z}_{>0}$ such that

$$(X, \Delta + H + \frac{2}{p^{d_a} - 1}(D_1 + \cdots + D_{\dim X}))$$

is sharply F -pure. In particular, we can find $e_a \in d_0 d_a \mathbb{Z}_{>0}$ such that the trace map

$$\begin{aligned} & \mathrm{Tr}_X^{e_a}((p^{e_a} - 1)(\Delta + H + \frac{2}{p^{d_a} - 1}(D_1 + \cdots + D_{\dim X})) \\ &= \mathrm{Tr}_{X \times \{a\}}^{e_a}((p^{e_a} - 1)(\Delta + H + \frac{2}{p^{d_a} - 1} \mathcal{D}|_{X \times \{a\}})) \end{aligned}$$

is surjective. By Proposition 8 and the properness of X , we can find an open subset $a \in U_a \subset T$ such that

$$\mathrm{Tr}_{X \times \{b\}}^{e_a}((p^{e_a} - 1)(\Delta + H + \frac{2}{p^{d_a} - 1} \mathcal{D}|_{X \times \{b\}}))$$

is surjective for every $\{b\} \in U_a \cap T(t, k_0)$. In particular, $(X \times \{b\}, \Delta + H + \frac{2}{p^{d_a} - 1} \mathcal{D}|_{X \times \{b\}})$ is sharply F -pure and

$$\mathrm{Tr}_{X \times \{b\}}^{ee_a}((p^{ee_a} - 1)(\Delta + H) + 2\mathcal{D}|_{X \times \{b\}}))$$

is surjective for every $e \in \mathbb{Z}_{>0}$ and every $b \in U_a \cap T(t, k_0)$. Therefore, we obtain an open cover

$$T(t, k_0) \subset \bigcup_{a \in T(t, k_0)} U_a.$$

By the compactness of $T(t, k_0)$, we obtain $T(t, k_0) \subset \bigcup_{1 \leq i \leq s} U_{a_i}$. Set $e_0 := e_{a_1} \cdots e_{a_s}$. Then, we see that the trace map

$$\mathrm{Tr}_{X \times \{c\}}^{e_0}((p^{e_0} - 1)(\Delta + H) + 2\mathcal{D}|_{X \times \{c\}}))$$

is surjective for every $c \in T(t, k_0)$.

Since $|D|$ is base point free and $T(t, k_0)$ is dense in T , we can find effective Cartier divisors

$$D'_1, \dots, D'_{p^{e_0}-1} \sim D$$

which satisfy the following properties.

- $[D'_i] \in T_1(k) = \mathbb{A}_k^q(k)$ for every $1 \leq i \leq p^{e_0} - 1$.
- For every $\{i_1, \dots, i_{\dim X}\} \subset \{1, 2, \dots, p^{e_0} - 1\}$ with $i_1 < i_2 < \dots < i_{\dim X}$, the point $[D'_{i_1}, \dots, D'_{i_{\dim X}}] \in T(k)$ is contained in $T(t, k_0)$.
- $\bigcap_{j \in J} D'_j = \emptyset$ for every subset $J \subset \{1, 2, \dots, p^{e_0} - 1\}$ with $|J| = \dim X + 1$.

We obtain

$$D \sim_{\mathbb{Q}} \frac{1}{p^{e_0} - 1} (D'_1 + \dots + D'_{p^{e_0}-1}) =: D'$$

and

$$\mathrm{Tr}_X^{e_0}((p^{e_0} - 1)(\Delta + H + 2D'))$$

is surjective at every point. By [SS, Theorem 3.9], in order to show that the pair $(X, \Delta + D')$ is strongly F -regular, it is enough to prove the following two assertions.

- (a) $\mathrm{Tr}_X^{e_0}((p^{e_0} - 1)(\Delta + D') + (p^{e_0} - 1)H + (p^{e_0} - 1)D')$ is surjective.
- (b) $(X \setminus \mathrm{Supp}(H + D'), (\Delta + D')|_{X \setminus \mathrm{Supp}(H + D')})$ is globally F -regular.

The assertion (a) follows from

$$(p^{e_0} - 1)(\Delta + D') + (p^{e_0} - 1)H + (p^{e_0} - 1)D' = (p^{e_0} - 1)(\Delta + H + 2D').$$

The assertion (b) holds because $(X \setminus \mathrm{Supp} H, \Delta|_{X \setminus \mathrm{Supp} H})$ is globally F -regular. We are done. \square

The proof of Proposition 3 is almost all the same as the one of Proposition 2. For the sake of completeness, we give a proof of it.

Proof of Proposition 3. By enlarging coefficients of Δ , we may assume that Δ is reduced, that is, $\Delta = \lfloor \Delta \rfloor$. By replacing D with its some multiple, we may assume that D is a Cartier divisor such that $|D|$ is base point free. In particular, $\dim_k H^0(X, D) \geq 2$, otherwise $D \sim 0$ and the assertion is trivial.

Set $T_1 := \mathbb{A}_k^q = \mathbb{A}(H^0(X, D))$ (i.e. $q := \dim_k H^0(X, D)$) and

$$T := (T_1)^{\dim X} = \mathbb{A}_k^q \times_k \cdots \times_k \mathbb{A}_k^q \simeq \mathbb{A}_k^{q \dim X}.$$

We can find an effective Cartier divisor \mathcal{D} on $X \times_k T$ which satisfies the following properties:

- Each k -rational point $a \in (\mathbb{A}_k^r(k))^{\dim X} = T(k)$ corresponds to a pair $(D_1, \dots, D_{\dim X})$, where, for every i , we obtain $D_i \sim D$ is an effective Cartier divisor. In this case, we write $a = [D_1, \dots, D_{\dim X}]$.
- $\mathcal{D}|_{X \times_k \{[D_1, \dots, D_{\dim X}]\}} = D_1 + \dots + D_{\dim X}$.

Fix a coordinate $t = (t_1, \dots, t_{q \dim X})$ of T . Note that we have a subset $T(t, k_0) \subset T(k)$, which is dense in T (cf. Definition 6).

Since $|D|$ is base point free, we can find a non-empty open subset $T^0 \subset T$ such that, for every $[D_1, \dots, D_{\dim X}] \in T(k) \cap T^0$, the support $\text{Supp}(\sum D_i)$ does not contain any F -pure centers of (X, Δ) .

We show that there exists $e_0 \in \mathbb{Z}_{>0}$ such that, for every $[D_1, \dots, D_{\dim X}] \in T(t, k_0) \cap T^0$, the trace map

$$\text{Tr}_X^{e_0}((p^{e_0} - 1)\Delta + (D_1 + \dots + D_{\dim X}))$$

is surjective. Fix $a = [D_1, \dots, D_{\dim X}] \in T(t, k_0) \cap T^0$. Then, by [Schwede, Main Theorem], we can find $d_a \in \mathbb{Z}_{>0}$ such that $(X, \Delta + \frac{1}{p^{d_a}-1}(D_1 + \dots + D_{\dim X}))$ is sharply F -pure. Therefore, we can find $e_a \in d_a \mathbb{Z}_{>0}$ such that the trace map

$$\begin{aligned} & \text{Tr}_X^{e_a}((p^{e_a} - 1)(\Delta + \frac{1}{p^{d_a}-1}(D_1 + \dots + D_{\dim X}))) \\ &= \text{Tr}_{X \times_k \{a\}}^{e_a}((p^{e_a} - 1)(\Delta + \frac{1}{p^{d_a}-1}\mathcal{D}|_{X \times_k \{a\}})) \end{aligned}$$

is surjective at every point. By Proposition 8 and the properness of X , we can find an open subset $a \in U_a \subset T$ such that

$$\text{Tr}_{X \times_k \{a\}}^{e_a}((p^{e_a} - 1)(\Delta + \frac{1}{p^{d_a}-1}\mathcal{D}|_{X \times_k \{b\}}))$$

is surjective for every $b \in T(t, k_0)$. Therefore,

$$\text{Tr}_{X \times_k \{a\}}^{ee_a}((p^{ee_a} - 1)(\Delta + \frac{1}{p^{d_a}-1}\mathcal{D}|_{X \times_k \{b\}}))$$

is surjective for every $b \in T(t, k_0)$ and every $e \in \mathbb{Z}_{>0}$. By the compactness of $T(t, k_0)$, we can find a required e_0 .

Since $|D|$ is base point free and $T(t, k_0)$ is dense in T , we can find effective Cartier divisors

$$D'_1, \dots, D'_{p^{e_0}-1} \sim D$$

which satisfy the following properties.

- $[D'_i] \in T_1(k) = \mathbb{A}_k^q(k)$ for every $1 \leq i \leq p^{e_0} - 1$.

- For every $\{i_1, \dots, i_{\dim X}\} \subset \{1, 2, \dots, p^{e_0} - 1\}$ with $i_1 < i_2 < \dots < i_{\dim X}$, the point $[D'_{i_1}, \dots, D'_{i_{\dim X}}] \in T(k)$ is contained in $T(t, k_0) \cap T^0$.
- $\bigcap_{j \in J} D'_j = \emptyset$ for every subset $J \subset \{1, 2, \dots, p^{e_0} - 1\}$ with $|J| = \dim X + 1$.

We obtain

$$D \sim_{\mathbb{Q}} \frac{1}{p^{e_0} - 1} (D'_1 + \dots + D'_{p^{e_0} - 1}) =: D'$$

and

$$\mathrm{Tr}_X^{e_0}((p^{e_0} - 1)(\Delta + D'))$$

is surjective. Thus, $(X, \Delta + D')$ is sharply F -pure. \square

Proof of Theorem 1. First we show the assertion for the case $\mathbb{K} = \mathbb{Q}$. Since there exists a log resolution, we may assume that X is regular and Δ is simple normal crossing. By Proposition 3 (resp. Proposition 2), we can find $D' \sim_{\mathbb{Q}} D$ such that $(X, \Delta + D')$ is sharply F -pure (resp. strongly F -regular). In particular, it is log canonical (resp. klt) by [HW, Theorem 3.3].

Second we prove the assertion for the case $\mathbb{K} = \mathbb{R}$. We can write $D = \sum_{i=1}^r a_i D_i$ where $a_i \in \mathbb{R}_{>0}$ and D_i is a semi-ample Cartier divisor. By the induction on r , we may assume that $r = 1$. Thus, we obtain $D = a_1 D_1$ where $a_1 \in \mathbb{R}_{>0}$ and D_1 is a semi-ample Cartier divisor. By replacing a_1 with $\lceil a_1 \rceil$, we may assume that D is a semi-ample Cartier divisor. Since there exists a log resolution, we may assume that X is regular and Δ is simple normal crossing. By enlarging coefficients of Δ a little, we can reduce the problem to the case $\mathbb{K} = \mathbb{Q}$. \square

REFERENCES

- [BSTZ] M. Blickle, K. Schwede, S. Takagi, W. Zhang, *Discreteness and rationality of F -jumping numbers on singular varieties*, Math. Ann. **347** (2010), 917–949.
- [BST] M. Blickle, K. Schwede, K. Tucker, *F -singularities via alterations*, to appear in Amer. J. of Math.
- [CHMS] P. Cascini, C. D. Hacon, M. Mustașă, K. Schwede, *On the numerical dimension of pseudo-effective divisors in positive characteristic*, Amer. J. of Math., **136** (2014), no. 6, 1609–1628.
- [CTX] P. Cascini, H. Tanaka, C. Xu, *On base point freeness in positive characteristic*, to appear in Ann. Sci. Éc. Norm. Sup.
- [HX] C. D. Hacon, C. Xu, *On the three dimensional minimal model program in positive characteristic*, to appear in J. Amer. Math. Soc.
- [HW] N. Hara, K. Watanabe, *F -regular and F -pure rings vs. log terminal and log canonical singularities*, J. Alg. Geom. **11**, 363–392.
- [Kollár] J. Kollár, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, vol. **200**, 2013.

- [Mustață] M. Mustață, *The non-nef locus in positive characteristic*, A celebration of algebraic geometry, 535–551, Clay Math. Proc., **18**, Amer. Math. Soc., Providence, RI, 2013.
- [PSZ] Z. Patakfalvi, K. Schwede, W. Zhang, *F-singularities in families*, preprint available at <http://arxiv.org/pdf/1305.1646v1.pdf>
- [Poonen] B. Poonen, *Bertini theorems over finite fields*, Ann. of Math., **160** (2004), 1099–1127.
- [Schwede] K. Schwede, *F-adjunction*, Algebra and Number Theory, Vol. **3**, no. 8, 907–950 (2009).
- [SS] K. Schwede, K. E. Smith, *Globally F-regular and log Fano varieties*, Advances in Mathematics, Vol. **224**, Issue 3, 863–894, 2010.
- [SZ] K. Schwede, W. Zhang, *Bertini theorems for F-singularities*, to appear in the Proceedings of the London Mathematical Society.
- [T] H. Tanaka, *The trace map of Frobenius and extending sections for three-folds*, to appear in Michigan Math. Journal.

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE, LONDON, 180 QUEEN'S GATE, LONDON SW7 2AZ, UK

E-mail address: `h.tanaka@imperial.ac.uk`